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P-adic Theta Functions and Solutions of the KP Hierarchy

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Abstract: Based on Schottky uniformization theory of Riemann surfaces, we construct a universal power series for (Riemann) theta function solutions of the KP hierarchy. Specializing this power series to the coordinates associated with Schottky groups over p-adic fields, we show that the p-adic theta functions of Mumford curves give solutions of the KP hierarchy.

Introduction

The KP (Kadomtsev–Petvyashvili) hierarchy is a system of infinitely many Lax type partial differential equations,

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L]; \qquad L = \partial_x + \sum_{i=1}^{\infty} u_i(x, t_1, t_2, \ldots)\partial_x^{-i}$$

($\partial_x = \partial/\partial x, (L^n)_+$: the nonnegative part of L^n for ∂_x) which includes the KP equation

$$\frac{3}{4}\frac{\partial^2 u_1}{\partial t_2^2} - \frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial t_3} - \frac{1}{4}\frac{\partial^3 u_1}{\partial x^3} - 3u_1\frac{\partial u_1}{\partial x}\right) = 0 .$$

Our final goal in this paper is to show that the p-adic theta functions of p-adic (Mumford) curves give solutions of the KP hierarchy. These solutions are included in algebro-geometric solutions for p-adic curves constructed by Krichever [Kr]. However, one cannot express these Krichever solutions in terms of p-adic theta functions as is done in [Kr] for the complex case because there is no theory on curvilinear integrals in p-adic analysis.

Our construction of p-adic solutions of the KP hierarchy consists of 2 steps: the first step is to obtain a "universal" solution expressed by a formal theta function, and the second step is to specialize this universal solution to p-adic solutions. For example, in the genus 1 case, the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{u\in L-\{0\}}\left(\frac{1}{(z-u)^2} - \frac{1}{u^2}\right) \quad (L := \mathbf{Z}(\pi\tau) + \mathbf{Z}\pi)$$