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Arithmetic Properties of Mirror Map and Quantum Coupling

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Abstract: We study some arithmetic properties of the mirror maps and the quantum Yukawa couplings for some 1-parameter deformations of Calabi–Yau manifolds. First we use the Schwarzian differential equation, which we derived previously, to characterize the mirror map in each case. For algebraic K3 surfaces, we solve the equation in terms of the J-function. By deriving explicit modular relations we prove that some K3 mirror maps are algebraic over the genus zero function field $\mathbf{Q}(J)$. This leads to a uniform proof that those mirror maps have integral Fourier coefficients. Regarding the maps as Riemann mappings, we prove that they are genus zero functions. By virtue of the Conway–Norton conjecture (proved by Borcherds using Frenkel–Lepowsky–Meurman's Moonshine module), we find that these maps are actually the reciprocals of the Thompson series for certain conjugacy classes in the Griess–Fischer group. This also gives, as an immediate consequence, a second proof that those mirror maps are integral. We thus conjecture a surprising connection between K3 mirror maps and the Thompson series. For threefolds, we construct a formal nonlinear ODE for the quantum coupling reduced mod p. Under the mirror hypothesis and an integrality assumption, we derive mod p congruences for the Fourier coefficients. For the quintics, we deduce, (at least for $5 \nmid d$) that the degree d instanton numbers n_d are divisible by 5^3 – a fact first conjectured by Clemens.

1. Introduction

For background on Mirror Symmetry, the readers are referred to reference [1, 2] (see especially the articles therein by Greene–Plesser, Candelas-de la Ossa–Green–Parkes. Katz, Morrison, Vafa and Witten).

It is known that the so-called mirror map and the quantum coupling have many interesting number theoretic properties based on numerical experiments – as previously observed by many [1, 3, 4, 5]. For example the Fourier coefficients of the

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