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Conformal Blocks on Elliptic Curves and the Knizhnik–Zamolodchikov–Bernard Equations

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Abstract: We give an explicit description of the vector bundle of WZW conformal blocks on elliptic curves with marked points as a subbundle of a vector bundle of Weyl group invariant vector valued theta functions on a Cartan subalgebra. We give a partly conjectural characterization of this subbundle in terms of certain vanishing conditions on affine hyperplanes. In some cases, explicit calculations are possible and confirm the conjecture. The Friedan–Shenker flat connection is calculated, and it is shown that horizontal sections are solutions of Bernard's generalization of the Knizhnik–Zamolodchikov equation.

1. Introduction

The aim of this work is to give a description of conformal blocks of the Wess–Zumino–Witten model on genus one curves as explicit as on the Riemann sphere.

Let us recall the well-known situation on the sphere. One fixes a simple finite dimensional complex Lie algebra $\mathfrak{g}$, with invariant bilinear form $(\,,)$ normalized so that the longest roots have length squared 2, and a positive integer k called level. One then considers the corresponding affine Kac–Moody Lie algebra, the one dimensional central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}((t))$ associated to the 2-cocycle $c(X \otimes f, Y \otimes g) = (X, Y)$ res $df g$. Its irreducible highest weight integrable representations of level (= value of central generator) k are in one to one correspondence with a certain finite set I_k of finite dimensional irreducible representations of $\mathfrak{g}$. These representations extend, by the Sugawara construction, to representations of the affine algebra to which an element L_{-1} is adjoined, such that $[L_{-1}, X \otimes f] = -X \otimes \frac{d}{dt} f$. Then to each n-tuple of distinct points $z_1, \dots, z_n$ on the complex plane, and of representations $V_1, \dots, V_n$ in I_k one associates the space of conformal blocks $E(z_1, \dots, z_n)$. It is the space of linear forms on the tensor product $\bigotimes_1^n V_i^\wedge$ of the corresponding level k representations of the affine algebra, which are annihilated by the Lie algebra $\mathscr{L}(z_1, \dots, z_n)$ of $\mathfrak{g}$-valued meromorphic functions with poles in $\{z_1, \dots, z_n\}$ and regular at infinity. The latter algebra acts on $\otimes V_i^\wedge$ by

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