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Quantization of a Class of Piecewise Affine Transformations on the Torus

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Abstract: We present a unified framework for the quantization of a family of discrete dynamical systems of varying degrees of "chaoticity." The systems to be quantized are piecewise affine maps on the two-torus, viewed as phase space, and include the automorphisms, translations and skew translations. We then treat some discontinuous transformations such as the Baker map and the sawtooth-like maps. Our approach extends some ideas from geometric quantization and it is both conceptually and calculationally simple.

1. Introduction

Interest in the quantization of discrete dynamical systems on compact phase spaces comes from the desire to understand the possible signature of classical chaotic dynamics in quantum mechanics. Recall for example that it is expected and in some cases proved that the asymptotic properties ($\hbar \to 0$) of the eigenfunctions of quantized systems depend on the degree of "chaoticity" of the corresponding classical ones (see, for instance, [Sar] and references therein). The torus forms an excellent testing ground for these ideas. Indeed, the simplest ergodic systems are the irrational translations on the torus, whereas the simplest hyperbolic dynamical systems are certain area-preserving maps [AA, CFS]. Among these, the best known are the toral automorphisms, the Baker transformation and some discontinuous maps such as the sawtooth map considered in [Ch, LW, V, Li]. It has been shown there that their singularities do not destroy the ergodicity and mixing properties one expects for hyperbolic maps.

One way in which the classical singularities will show up at the quantum level is as follows. For the linear automorphisms the classical and the quantum evolution are identical, as in the harmonic oscillator. The singularities will destroy this property,

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