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Scalar Conservation Laws with Discontinuous Flux Function: II. On the Stability of the Viscous Profiles

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Abstract: The equation $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(H(x)f(u) + \left(1 - H(x)\right)g(u)\right) = 0$, where H is Heaviside's step function, appears for example in continuous sedimentation of solid particles in a liquid, in two-phase flow, in traffic-flow analysis and in ion etching. The discontinuity of the flux function at $x = 0$ causes a discontinuity of a solution, which is not uniquely determined by the initial data. By a viscous profile of this discontinuity we mean a stationary solution of $u_t + (F^\delta)_x = \varepsilon u_{xx}$, where F^δ is a smooth approximation of the discontinuous flux, i.e., H is smoothed. We present some results on the stability of the viscous profiles, which means that a small disturbance tends to zero uniformly as $t \to \infty$. This is done by weighted energy methods, where the weight (depending on f and g) plays a crucial role.

1. Introduction

The scalar conservation law with discontinuous flux function

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\Big(F^0\big(u(x,t),x\big)\Big) = 0, \text{ where } F^0(u,x) = \begin{cases} f(u), & x > 0 \\ g(u), & x < 0 \end{cases} \tag{1.1}$$

arises in several applications, for example in continuous sedimentation of solid particles in a liquid, see Diehl [4] and Chancelier et al. [1], in two-phase flow in porous media, see Gimse and Risebro [5], in traffic-flow analysis, see Mochon [15], and in ion etching in the fabrication of semiconductor devices, see Ross [17]. The Cauchy problem for a more general equation than (1.1), including a point source $s(t)$ at $x = 0$, has been analysed by Diehl [3]. Generally, solutions of (1.1) contain a discontinuity along the t-axis and curves of discontinuity that go into and emanate from it. This discontinuity along the t-axis is not uniquely determined by the initial data $u(x, 0)$ and a uniqueness condition, Condition Γ, was introduced in [3]. Another way to pick out the physically relevant discontinuity is by means of the viscous profile condition. By a viscous profile we mean a stationary solution of $u_t + (F^\delta)_x = \varepsilon u_{xx}$, where F^δ is a smooth approximation of the discontinuous flux, i.e., H is smoothed. In [2] the equivalence between Condition Γ and the viscous profile condition is presented, as