Commun. Math. Phys. 176, 23–44 (1996)

Communications in
Mathematical
Physics

Scalar Conservation Laws with Discontinuous Flux Function: I. The Viscous Profile Condition

S. Diehl

Department of Mathematics, Lund Institute of Technology, P.O. Box 118, S-221 00 Lund, Sweden, E-mail: diehl@maths.lth.se

Received: 22 November 1994

Abstract: The equation $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(H(x)f(u) + \left(1 - H(x)\right)g(u)\right) = 0$, where H is Heaviside's step function, appears for example in continuous sedimentation of solid particles in a liquid, in two-phase flow, in traffic-flow analysis and in ion etching. The discontinuity of the flux function at $x = 0$ causes a discontinuity of a solution, which is not uniquely determined by the initial data. The equation can be written as a triangular 2×2 non-strictly hyperbolic system. This augmentation is non-unique and a natural definition is given by means of viscous profiles. By a viscous profile we mean a stationary solution of $u_t + (F^\delta)_x = \varepsilon u_{xx}$, where F^δ is a smooth approximation of the discontinuous flux, i.e., H is smoothed. In terms of the 2×2 system, the discontinuity at $x = 0$ is either a regular Lax, an under- or overcompressive, a marginal under- or overcompressive or a degenerate shock wave. In some cases, depending on f and g, there is a unique viscous profile (e.g. undercompressive and regular Lax waves) and in some cases there are infinitely many (e.g. overcompressive waves). The main purpose of the paper is to show the equivalence between a previously introduced uniqueness condition for the discontinuity of the solution at $x = 0$ and the viscous profile condition.

1. Introduction

The scalar conservation law with discontinuous flux function

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\Big(F^0\big(u(x,t),x\big)\Big) = 0, \quad \text{where } F^0(u,x) = \begin{cases} f(u), & x > 0 \\ g(u), & x < 0 \end{cases} \tag{1.1}$$

arises in several applications, for example in continuous sedimentation of solid particles in a liquid, see Diehl [3] and Chancelier et al. [1], in two-phase flow in porous media, see Gimse and Risebro [6], in traffic-flow analysis, see Mochon [11], and in ion etching in the fabrication of semiconductor devices, see Ross [13]. In these applications (except perhaps in traffic flow) the flux functions f and g are non-convex. The Cauchy problem for a more general equation than (1.1), including a point source $s(t)$ at $x = 0$, has been analysed by the author in [2]. Generally, a solution of (1.1)