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Ergodicity of Eigenfunctions for Ergodic Billiards

Steven Zelditch, Maciej Zworski
Mathematics Department, Johns Hopkins University, Baltimore, MD21212-2689, USA

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Abstract: We give a simple proof of ergodicity of eigenfunctions of the Laplacian with Dirichlet boundary conditions on compact Riemannian manifolds with piecewise smooth boundaries and ergodic billiards. Examples include the "Bunimovich stadium", the "Sinai billiard" and the generic polygonal billiard tables of Kerckhoff, Masur and Smillie.

1. Introduction and Statement of Results

The purpose of this note is to give a simple proof of ergodicity of eigenfunctions on manifolds with piecewise smooth boundaries and ergodic billiard flows. For the boundaryless case this is now well known and was established in [14, 17, 3]. If the manifold is a convex region in $\mathbb{R}^n$ with $W^{2,\infty}$ boundary, the same result was proved much more recently by Gérard and Leichtnam [6] who needed, among other things, a quantitative version of the results of Melrose and Sjöstrand on propagation of singularities for mixed problems (see [12 and 9, Chap. 24]). Our argument uses an aspect of an abstract approach to quantum ergodicity developed by the first author in [18] and some basic microlocal analysis on manifolds with boundary. It applies to a class of (not necessarily convex) regions with measure theoretically negligible sets of singular points. That class includes the much studied example of the "Bunimovich stadium" and other examples of non-dispersive billiard tables [1], the "Sinai billiard" [15] as well as the ergodic polygonal billiard tables of Kerckhoff, Masur and Smillie [11] – see Fig. 1.

Referring to Sect. 2 for precise definitions of intuitively clear notions of manifolds with piecewise smooth boundaries and of billiard flows, we can state our result as

Theorem. *Let (M, g) be a compact $\mathscr{C}^\infty$ Riemann manifold with piecewise smooth boundary. Let $\{\phi_j\}_{j\in\mathbb{N}_0}$ be an orthonormal set of eigenfunctions of the Laplacian, Δ_g, on M. If the billiard flow defined on a set of full measure in S^*M is ergodic then there exists a subsequence $\{j_k\}$ of density $1, \#\{j_k : j_k \leqq N\}/N \to 1$*