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The Variational Problems for Classical N-Vector Models

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Abstract: We prove existence and regularity properties of solutions of the variational problems introduced in the previous paper [1] for classical lattice N-vector models. These results form a basis of our renormalization group approach to low temperature expansions for the considered models.

1. Introduction

In this paper we study the basic variational problems for the classical spin models introduced in [1], like the problems (2.1), (2.2), (2.6), (2.7) there. We do not want to study these entirely small field cases because it is almost equally simple to consider a general case. Thus we study here the variational problems in a form appearing in a general case involving large field domains also. To formulate them we need some additional definitions. At first we have to determine a geometric setting of the problems. We consider a sequence of domains $\{\Omega_j\}$, $\Omega_j \subset T_\eta$, connected components of Ω_j belong to $\mathscr{D}_j$, $j = 1, 2, \ldots, k$ such that

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_k, \quad (L^j\eta)^{-1} \text{ dist } (\Omega_j^c,\ \Omega_{j+1}) > RM\,, \tag{1.1}$$

where R is a positive integer which will be fixed later. Let us recall that the lattice T_η and the classes $\mathscr{D}_j$ of localization domains have been introduced in Sect. 1 and 2 of [1]. The size of big blocks M here may not be equal to the one there. In this paper we obtain conditions on M connected with the variational problems, but there will be other conditions in the following papers, so we treat it as one of adjustable parameters. We define

$$\Lambda_j = \Omega_j^{(j)} \backslash \Omega_{j+1}^{(j)}, \quad j = 1, 2, \ldots, k-1, \quad \Lambda_k = \Omega_k^{(k)}, \quad \Lambda_0 = \Omega_1^c \cap \Omega_1^{\sim}\,,$$

$$\text{where } \Omega_j^{(i)} = \Omega_j \cap T_{L^i\eta}^{(i)} \text{ is the set of centers of } i\text{-blocks}$$

$$\text{in } \Omega_j, \quad \text{hence } \Omega_j = B^i(\Omega_j^{(i)}), \quad \text{and} \quad \Omega_j = \bigcup_{n=j}^{k} B^n(\Lambda_n)\,. \tag{1.2}$$

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