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Quantum Discrete Sine-Gordon Model at Roots of 1: Integrable Quantum System on the Integrable Classical Background

V. Bazhanov[1], A. Bobenko[2], N. Reshetikhin[3]

[1] Mathematics and Theoretical Physics, IAS, Australian National University, Canberra, ACT 2601, Australia
[2] Fachbereich Mathematik, Technische Universität Berlin, D-10623 Berlin, Germany
[3] Department of Mathematics, University of California, Berkeley, California 94720, USA

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Dedicated to L.D. Faddeev on his 60th birthday

Abstract: The quantum discrete sine-Gordon model at roots of 1 is studied. It is shown that this model provides an example of an integrable quantum system in an integrable classical background. In particular, the spectrum of quantum integrals of motions in this model depends only on the values of integrals of motion of a background classical system.

1. Introduction

The sine-Gordon equation is a nonlinear differential equation for a scalar function ϕ of two variables:

$$-\partial_t^2\phi + \partial_x^2\phi = 4\sin\phi\,. \tag{1.1}$$

The Cauchy problem for this equation with initial data $\phi(x,t)|_{t=0} = \phi(x), \partial_t\phi(x,t)|_{t=0} = \pi(x)$ can be regarded as an infinite-dimensional Hamiltonian mechanical system. The functions $(\pi(x), \phi(x))$ are "natural canonical coordinate functions" on the phase space of this system with Poisson brackets:

$$\{\pi(x), \phi(y)\} = \delta(x-y)\,. \tag{1.2}$$

The Hamiltonian which generates evolution (1.1) on the phase space with the Poisson structure (1.2) is

$$\mathscr{H} = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{2}\pi(x)^2 + \frac{1}{2}(\partial_x\phi(x))^2 + 4(1-\cos\phi(x))\right) dx\,, \tag{1.3}$$

where we assume the convergence of the integral.

The Hamiltonian system (1.2), (1.3) is integrable. See [FT] for more complete description of the sine-Gordon system.

The quantization of this model has been done in several steps. For a quasiclassical analysis, which includes the quantization of solitons in Eq. (1.1), see [FK]. A phenomenological scattering theory with a factorized S-matrix has been constructed in [ZZ]. The Bethe-ansatz solution has been found in [FST].