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Variational Characterization of the Speed of Propagation of Fronts for the Nonlinear Diffusion Equation

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Abstract: We give an integral variational characterization for the speed of fronts of the nonlinear diffusion equation $u_t = u_{xx} + f(u)$ with $f(0) = f(1) = 0$, and $f > 0$ in $(0,1)$, which permits, in principle, the calculation of the exact speed for arbitrary f.

1. Introduction

The problem of the asymptotic speed of propagation of the interface between an unstable and stable state has received much attention in connection with different problems of population growth, chemical reactions, pattern formation and others. We refer to [1] for a recent review and references. The best understood of such problems is that of the nonlinear reaction diffusion equation

$$u_t = u_{xx} + f(u) \tag{1a}$$

with

$$f(0) = f(1) = 0, \quad f'(0) > 0 \quad \text{and} \quad f > 0 \quad \text{in } (0,1) \tag{1b}$$

for which Aronson and Weinberger [2] have shown that any positive sufficiently localized initial condition $u(x,0)$ evolves into a front that joins the stable state $u = 1$ to $u = 0$. The asymptotic speed at which the front propagates is the minimal speed c^* for which there is a monotonic front joining $u = 1$ to $u = 0$. Moreover they show that the selected speed is bounded above and below by

$$2\sqrt{f'(0)} \leqq c^* \leqq 2 \sup \left\{ \sqrt{\frac{f(u)}{u}} \,\middle|\, u \in (0,1) \right\} \tag{2}$$

and that the asymptotic selected front approaches the fixed point $u = 0$ exponentially with slope

$$m = -\frac{1}{2}(c^* + \sqrt{c^{*2} - 4f'(0)})\,. \tag{3}$$