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Functional Equation for Dynamical Zeta Functions of Milnor–Thurston Type

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Abstract: A Milnor–Thurston type dynamical zeta function $\zeta_L(Z)$ is associated with a family of maps of the interval $(-1, 1)$. Changing the direction of time produces a new zeta function $\zeta'_L(Z)$. These zeta functions satisfy a functional equation $\zeta_L(Z)\zeta'_L(\varepsilon Z) = \zeta_0(Z)$ (where ε amounts to sign changes and, generically, $\zeta_0 \equiv 1$). The functional equation has non-trivial implications for the analytic properties of $\zeta_L(Z)$.

0. Introduction

Milnor and Thurston [2] have shown how the zeta function $\zeta(z)$ counting the periodic points of a piecewise monotone interval map f could be expressed in terms of a *kneading determinant* $D(z)$. The zeta function considered by Milnor and Thurston is closely related to the Lefschetz zeta function ζ_L, which we shall use henceforth. Baladi and Ruelle [1] have shown how to replace z in the Milnor–Thurston formula by $Z = (z_1, \ldots, z_N)$, where the interval of definition of f is cut into subintervals with different weights z_i. We shall here use a further extension of the formula $\zeta_L(Z) = D(Z)$, where f is allowed to be multivalued. The inverse f^{-1} of f is again multivalued piecewise monotone; it is associated with a zeta function $\zeta'_L(Z)$. There is a natural relation (*functional equation*)

$$\zeta_L(Z)\zeta'_L(\varepsilon Z) = \zeta_0(Z),$$

where ε corresponds to some sign changes and $\zeta_0(Z)$ counts "exceptional" orbits (generically $\zeta_0(Z) = 1$). The analytic properties of $\zeta_L(Z)$ are related, via the kneading determinant $D(Z)$, to the spectral properties of a *transfer operator* $\mathcal{M}_Z$. The spectral properties needed here are a refinement of those proved in Ruelle [4]. Using these properties one shows that ζ_L is meromorphic in a certain domain, with poles only if 1 is an eigenvalue of $\mathcal{M}_Z$. Let $\mathcal{M}'_Z$ denote the transfer operator corresponding to f^{-1}; using the functional equation one shows that ζ_L can vanish only if 1 is an eigenvalue of $\mathcal{M}'_Z$.

In what follows we shall write ζ instead of ζ_L, and use a family (ψ_ω) of monotone maps, instead of the multivalued map f^{-1}. *Warning*: If the ψ_ω are the branches