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Non-Commutative Chaotic Expansion of Hilbert–Schmidt Operators on Fock Space

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Abstract: It is known, from a simple algebraic computation, that every Hilbert–Schmidt operator on the Fock space admits a Maassen–Meyer kernel. Maassen–Meyer kernels are a non-commutative extension of the usual notion of chaotic expansion of random variables. Using an extension of the non-commutative stochastic integrals which allows to define these integrals on the whole Fock space, we prove that a Hilbert–Schmidt operator on Fock space is the sum of a series of iterated non-commutative stochastic integrals with respect to the basic three quantum noises. In this way we recover its Maassen–Meyer kernel which can be completely described from the operator itself.

1. Introduction

It is well-known that every square integrable functional f of the Wiener process $(W_t)_{t\geqq 0}$ admits a *previsible representation*, that is a representation as the sum of a constant (its expectation) and a stochastic integral of a previsible process with respect to W. But such a random variable also admits a *chaotic expansion* [7], that is, a representation of the form

$$f = \mathbb{E}[f] + \sum_{n=1}^{\infty} \int_{0<t_1<\cdots<t_n} f_n(t_1,\ldots,t_n)\, dW_{t_1}\cdots dW_{t_n}\,,$$

where f_n is a square integrable function on the increasing simplex

$$\Sigma_n = \{(t_1,\ldots,t_n) \in \mathbb{R}^n,\ 0 < t_1 < \cdots < t_n\}\,.$$

The set $\mathscr{P}_n$ of subsets of $\mathbb{R}^+$ with cardinality n can be clearly identified to Σ_n. The family $\{f_n\}$ can be viewed as a single square integrable function $\widehat{f}$ on $\mathscr{P} = \bigcup_n \mathscr{P}_n (\mathscr{P}_0 = \{\emptyset\})$, by putting $\widehat{f}(A) = f_n(t_1,\ldots,t_n)$ if $A = \{0 < t_1 < \cdots < t_n\} \in \mathscr{P}$, with the convention $\widehat{f}(\emptyset) = \mathbb{E}[f]$. With this "short notation" ([3]) the chaotic expansion of f can be written $f = \int_{\mathscr{P}} \widehat{f}(A)\, dW_A$.

On the boson Fock space Φ over $L^2(\mathbb{R}^+)$, which is isomorphic to the space of square integrable Wiener functionals ([15]), *operators* can be represented in two