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Chaotic Properties of the Elliptical Stadium

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Abstract: The elliptical stadium is a curve constructed by joining two half-ellipses, with half axes $a > 1$ and $b = 1$, by two straight segments of equal length $2h$.

Donnay [6] has shown that if $1 < a < \sqrt{2}$ and if h is big enough, then the corresponding billiard map has a positive Lyapunov exponent almost everywhere; moreover, $h \to \infty$ as $a \to \sqrt{2}$.

In this work we prove that if $1 < a < \sqrt{4 - 2\sqrt{2}}$, then $h > 2a^2\sqrt{a^2 - 1}$ assures the positiveness of a Lyapunov exponent. And we conclude that, for these values of a and h, the elliptical stadium billiard mapping is ergodic and has the K-property.

1. Introduction

The plane billiard consists in the free motion of a point particle on a connected bounded region in $\mathscr{R}^2$, being reflected elastically at the boundary. The billiard defines a 2-dimensional discrete dynamical system.

Depending on the boundary, this discrete dynamical system may have different dynamical behaviour. For instance, if it is a circle or an ellipse, it is known that the system is integrable, and the phase space is ordered by invariant curves (Fig. 1).

A quite different situation appears when the components of the boundary have negative curvature. The system is, then, ergodic and almost every orbit is dense on the phase space (as in Sinai billiards).

The first example of an ergodic billiard with convex boundary was given by Bunimovich [3]. The boundary is the circular stadium, composed by joining two half-circles by means of two straight segments of equal length.

A convex generalization is the elliptical stadium, composed by two half-ellipses, with half axes $a \geqq 1$ and $b = 1$, joined by two straight segments of equal length $2h$. For $a = 1$, i.e., for the circular stadium billiard, $h > 0$ implies the existence

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