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Convex Delay Endomorphisms

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Abstract: In this paper delay equations $x_{n+k} = f(x_n,\dots,x_{n+k-1})$ are considered, where the function f is supposed to be convex, having a unique point of maximum. It is proved that if there are no stationary solutions then all solutions must diverge. Considering the one parameter family $f_\mu = \mu + f$ and associating to it a family of two dimensional maps F_μ it is shown that the set of points having bounded orbit under F_μ is homeomorphic to the product of a Cantor set and a circle, and is hyperbolic and stable.

1. Introduction

Any delay equation of order k:

$$x_{n+k} = f(x_n,\dots,x_{n+k-1}) \tag{1}$$

can be associated with a transformation of R^k given by

$$F(x_1,\dots,x_k) = (x_2,\dots,x_k, f(x_1,\dots x_k)) . \tag{2}$$

Any orbit of the map F is in one to one correspondence with a solution of the delay equation (1). Here we will deal with delay equations where the function f is *convex*, in the sense that f is a C^2 function such that the quadratic form associated with the second derivative is definite at every point. In this case Eq. (1) is called a convex delay equation and the map F defined in (2) is called a convex delay endomorphism. In the rest of this work, we will take this quadratic form negatively definite, so that f could have at most one critical point that should be a maximum. A stationary solution of the delay equation (1) is a constant solution $x_n = x$ for every n; the existence of such an x is equivalent to have a solution of the equation $f(x,\dots,x) = x$. Moreover, the fixed points of F are the points $(x,\dots,x)$, where x is a solution of $f(x,\dots,x) = x$. So when f is convex the delay equation associated would have at most two stationary solutions, or, which is the same, the