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Reduction of Constrained Mechanical Systems and Stability of Relative Equilibria

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Abstract: A mechanical system with perfect constraints can be described, under some mild assumptions, as a constrained Hamiltonian system (M, Ω, H, D, W): (M, Ω) (the *phase space*) is a symplectic manifold, H (the *Hamiltonian*) a smooth function on M, D (the *constraint submanifold*) a submanifold of M, and W (the *projection bundle*) a vector sub-bundle of $T_D M$, the reduced tangent bundle along D. We prove that when these data satisfy some suitable conditions, the time evolution of the system is governed by a well defined differential equation on D. We define constrained Hamiltonian systems with symmetry, and prove a reduction theorem. Application of that theorem is illustrated on the example of a convex heavy body rolling without slipping on a horizontal plane. Two other simple examples show that constrained mechanical systems with symmetry may have an attractive (or repulsive) set of relative equilibria.

1. Introduction

Let (M, Ω) be a symplectic manifold on which a Lie group G acts by a Hamiltonian action, with an equivariant momentum map J. Let μ be a regular (or weakly regular) value of J. Under some general assumptions, the very important concept of *symplectic reduction*, due to K. Meyer [24], J. Marsden and A. Weinstein [22], allows us to obtain from these data a new symplectic manifold $(P_\mu = J^{-1}(\mu)/G_\mu, \Omega_P)$, called the *reduced symplectic manifold* at μ. Any smooth G-invariant Hamiltonian H on M induces a smooth reduced Hamiltonian H_P on P_μ, and the integral curves of the Hamiltonian vector field X_H associated with H contained in $J^{-1}(\mu)$ project onto the integral curves of the Hamiltonian vector field X_{H_P} associated with H_P. In particular, relative equilibria of X_H contained in $J^{-1}(\mu)$ project onto equilibria of X_{H_P}, i.e., onto points on P_μ where X_{H_P} vanishes. Stability properties of these relative equilibria are closely related to stability properties of the corresponding equilibria in P_μ.

Symplectic reduction plays an important part in symplectic geometry, in analytical mechanics and in mathematical physics. It has been extended to vari-