Commun. Math. Phys. 174, 161 – 190 (1995)

Communications in Mathematical Physics

Meromorphic Zeta Functions for Analytic Flows

David Fried★

Department of Mathematics, Boston University, Boston, MA 02215, USA

Received: 12 August 1994/in revised form: 3 January 1995

Dedicated to Steve Smale

Abstract: We extend to hyperbolic flows in all dimensions Rugh's results on the meromorphic continuation of dynamical zeta functions. In particular we show that the Ruelle zeta function of a negatively curved real analytic manifold extends to a meromorphic function on the complex plane.

In this paper we address a problem Smale poses in his survey article on dynamical systems ([Sm], II.4): given an isolated, compact, hyperbolic set Ω for a flow ϕ on a manifold M, find a meromorphic function on **C** which admits the product expansion

$$R(z) = \prod_{\gamma}(1 - e^{-z\ell(\gamma)})$$

for $\mathrm{Re}\, z \gg 0$, where γ runs over the periodic trajectories in Ω of multiplicity 1 and $\ell(\gamma)$ denotes the period of γ. The theorem in Sect. 7 does this for ϕ a C^ω (real analytic) flow.

In fact we meromorphically extend any "zeta function in one variable for (Ω, ϕ)" to **C**. A precise definition is given at the end of Sect. 6, with a discussion which shows how it includes all the usual examples. In particular we treat Selberg's zeta function $S(z)$ for a cocompact Fuchsian group Γ, where $\Omega = \Gamma\backslash PSl(2,\mathbf{R})$ and ϕ_t is given by right multiplication by the one parameter group diag $(e^{t/2}, e^{-t/2})$, and where $S(z)$ is defined as

$$S(z) = R(z)R(z+1)R(z+2)\dots .$$

In [Se], Selberg uses his trace formula to meromorphically extend $S(z)$. Hence $R(z) = S(z)/S(z+1)$ also has a meromorphic extension in this case, which motivated Smale's problem.

This paper combines the methods of a seminal paper of Ruelle with an innovative idea of Rugh. Ruelle's paper [R1] concerns the case where the stable and unstable bundles E^s, E^u of $\phi_t|\Omega$ extend to C^ω bundles on a neighborhood of Ω.

★ This work was supported in part by I.H.E.S. and the National Science Foundation.