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On the Total Bandwidth for the Rational Harper's Equation

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Abstract: In the last years several contributions have been done around the total bandwidth of the spectrum for the Harper's operator. In particular an interesting conjecture has been proposed by Thouless which gives also strong convincing arguments for the proof in special cases. On the other hand, in the study of the Cantor structure of the spectrum, B. Helffer and J. Sjöstrand have justified an heuristic semi-classical approach proposed by M. Wilkinson. The aim of this article is to explain how one can use the first step of this approach to give a rigorous semi-classical proof of the Thouless formula in some of the simplest cases. We shall also indicate how one can hope with more effort to prove rigorously recent results of Last and Wilkinson on the same conjecture.

1. Introduction

The Harper's operator $H_{\alpha,\lambda,\theta}$ is defined on $l^2(\mathbb{Z})$ as

$$(u_n)_{n\in\mathbb{Z}} \to (H_{\alpha,\lambda,\theta}u)_n = \frac{1}{2}(u_{n+1} + u_{n-1}) + \lambda \cos(2\pi\alpha n + \theta)u_n . \tag{1.1}$$

Here α and θ are real parameters satisfying:

$$0 \leqq \alpha \leqq 1, \quad 0 \leqq \theta \leqq 2\pi . \tag{1.2}$$

The most interesting spectral properties of this operator appear when α is irrational. In this case a Cantor structure for the spectrum is expected. As was observed by D. Hofstadter who get a beautiful butterfly [16], we get a good intuition of the problem by a careful analysis of the spectrum in the rational case. In this case:

$$\alpha = p/q$$

and the spectrum $\sigma(\alpha,\lambda,\theta)$ of the Harper's operator depends effectively on θ and one more interesting spectrum is the closed set:

$$\Sigma(\alpha,\lambda) =_{def} \bigcup_{\theta\in[0,2\pi]} \sigma(\alpha,\lambda,\theta) \tag{1.3}$$