Commun. Math. Phys. 173, 101 – 133 (1995)

Communications in Mathematical Physics

Caustics for Inner and Outer Billiards

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Received: 1 May 1994 / in revised form: 15 August 1994

Abstract: With a plane closed convex curve, T, we associate two area preserving twist maps: the (classical) inner billiard in T and the outer billiard in the exterior of T. The invariant circles of these twist maps correspond to certain plane curves: the inner and the outer caustics of T. We investigate how the shape of T determines the possible location of caustics, establish the existence of open regions which are free of caustics, and estimate from below the size of these regions in terms of the geometry of T.

Introduction

A closed convex curve, $T \subset \mathbf{R}^2$, in the Euclidean plane defines two natural dynamical systems: the classical billiard ball map, ϕ_T (inside T), and the "outer billiard," ψ_T, in $\mathbf{R}^2 \setminus \mathrm{int}(T)$. The phase space, $\Phi = \Phi_T$, of the inner billiard is the space of directed straight lines (rays) intersecting T. The phase space, Ψ_T, of the outer billiard map is the set of points in the exterior of T. With a natural choice of coordinates in the phase space, both billiards are area preserving twist maps.

Invariant circles for these twist maps correspond to certain geometric objects. In the case of ϕ_T these are the caustics of T (famous in the geometric optics). A caustic, $\gamma \subset \mathrm{int}(T)$, corresponding to an arbitrary invariant circle Γ may have a complicated structure. For instance, the jumps of the tangent direction for Γ correspond to the discontinuities of γ. In particular, these caustics are not convex. There are other types of nonconvex caustics. For instance, the equator of Φ_T, for a table T of constant width, is an invariant circle. The corresponding caustic is not convex (unless T is circular). In Sect. 1 where we deal with the inner billiard map, we restrict our setting to the convex caustics. The present techniques do not work without this assumption.

If a caustic γ is convex, then it is simply the envelope of the corresponding family, Γ, of rays. For sufficiently smooth tables T a necessary and sufficient

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⋆ Partially supported by NSF. email: ⟨egutkin@math.usc.edu⟩

⋆⋆ Partially supported by NSF Grant DMS 9017995.