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Invariants of 3-Manifolds and Projective Representations of Mapping Class Groups via Quantum Groups at Roots of Unity

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Abstract: An example of a finite dimensional factorizable ribbon Hopf $\mathbb{C}$-algebra is given by a quotient $H = u_q(\mathfrak{g})$ of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ at a root of unity q of odd degree. The mapping class group $M_{g,1}$ of a surface of genus g with one hole projectively acts by automorphisms in the H-module $H^{*\otimes g}$, if H^* is endowed with the coadjoint H-module structure. There exists a projective representation of the mapping class group $M_{g,n}$ of a surface of genus g with n holes labeled by finite dimensional H-modules $X_1, \dots, X_n$ in the vector space $\mathrm{Hom}_H(X_1 \otimes \cdots \otimes X_n, H^{*\otimes g})$. An invariant of closed oriented 3-manifolds is constructed. Modifications of these constructions for a class of ribbon Hopf algebras satisfying weaker conditions than factorizability (including most of $u_q(\mathfrak{g})$ at roots of unity q of even degree) are described.

After works of Moore and Seiberg [44], Witten [63], Reshetikhin and Turaev [51], Walker [62], Kohno [22, 23] and Turaev [60] it became clear that any semisimple abelian ribbon category with a finite number of simple objects satisfying some non-degeneracy condition gives rise to projective representations of mapping class groups of surfaces as well as to invariants of closed 3-manifolds. It was proposed in [38] to get rid of semisimplicity and to extend so the class of categories which serve as the set of labels for a modular functor.

In this article we describe (eventually non-semisimple) ribbon Hopf algebras H, whose modules form a category with the required properties, thereby giving representations of mapping class groups. These algebras are called 2-modular. All finite dimensional factorizable ribbon Hopf algebras have those properties.

As a byproduct we obtain a projective representation of the mapping class group $M_{g,1}$ of a surface of genus g with one hole in the vector space $H^{*\otimes g}$. If H^* is endowed with the coadjoint H-module structure, $M_{g,1}$ acts by automorphisms of the H-module. For genus 1 and factorizable Hopf algebras this representation was obtained by Majid and the author [40]. In the case of Drinfeld's doubles another proof of modular relations for genus 1 was given by Kerler [16]. The projective

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