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Homoclinic Orbits on Compact Hypersurfaces in $\mathbb{R}^{2N}$, of Restricted Contact Type

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Abstract: Consider a smooth Hamiltonian system in $\mathbb{R}^{2N}$, $\dot{x} = JH'(x)$, the energy surface $\Sigma = \{x/H(x) = H(0)\}$ being compact, and 0 being a hyperbolic equilibrium. We assume, moreover, that $\Sigma\backslash\{0\}$ is of restricted contact type. These conditions are symplectically invariant. By a variational method, we prove the existence of an orbit homoclinic, i.e. non-constant and doubly asymptotic, to 0.

I. Introduction

The goal of this work is to give a partial answer to a conjecture of Helmut Hofer, about homoclinic orbits in Hamiltonian systems (personal communication). Suppose that Σ is the zero energy surface of an autonomous Hamiltonian H in $\mathbb{R}^{2N}$ having $x_0 \in \Sigma$ as a hyperbolic equilibrium and no other equilibrium on Σ, and that $\Sigma\backslash\{x_0\}$ is of contact type. These conditions are symplectically invariant. The conjecture is that the Hamiltonian system

$$\dot{x} = X_H\,, \quad X_H = JH'(x)\,, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{1.1}$$

has at least one solution $x(t)$ homoclinic to x_0, i.e. such that $x \not\equiv x_0$, and $\lim_{|t|\to\infty} x(t) = x_0$. It may be seen as an analogue for homoclinic orbits of the Weinstein conjecture in $\mathbb{R}^{2N}$, which was solved by Viterbo in 1987 (see [W, V, H-Z]). In the present paper, we replace the contact condition by a restricted contact condition, less general but also symplectically invariant. We find a homoclinic orbit, as the critical point of the action functional associated to a suitably chosen Hamiltonian.

More precisely, we consider the following set of hypotheses on Σ:

($\mathscr{H}1$): Σ is a compact set. It may be defined as $\Sigma = \{x/H(x) = 0\}$, where H is a smooth Hamiltonian defined on $\mathbb{R}^{2N}$, whose differential H' does not vanish on Σ, except at one point x_0 that we identify with 0 after translation. Moreover, $A = H''(0)$ is non-degenerate.

($\mathscr{H}2$): JA is hyperbolic, i.e. $sp(JA) \cap i\mathbb{R} = \emptyset$.