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Classical and Quantum Integrable Systems in $\widetilde{\mathfrak{gl}}(2)^{+*}$ and Separation of Variables

J. Harnad[1,2] **P. Winternitz**[2]

[1] Department of Mathematics and Statistics, Concordia University, 7141, Sherbrooke W., Montréal Québec, Canada H4B 1R6
e-mail address: harnad@alcor.concordia.ca, harnad@mathcn.umontreal.ca
[2] Centre de recherches mathématiques, Université de Montréal, C.P. 6128-A, Montréal, Québec, Canada H3C 3J7 e-mail address: wintern@ere.umontreal.ca

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Abstract: Classical integrable Hamiltonian systems generated by elements of the Poisson commuting ring of spectral invariants on rational coadjoint orbits of the loop algebra $\widetilde{\mathfrak{gl}}^{+*}(2, \mathbb{R})$ are integrated by separation of variables in the Hamilton–Jacobi equation in hyperellipsoidal coordinates. The canonically quantized systems are then shown to also be completely integrable and separable within the same coordinates. Pairs of second class constraints defining reduced phase spaces are implemented in the quantized systems by choosing one constraint as an invariant, and interpreting the other as determining a quotient (i.e. by treating one as a first class constraint and the other as a gauge condition). Completely integrable, separable systems on spheres and ellipsoids result, but those on ellipsoids require a further modification of order $\mathcal{O}(\hbar^2)$ in the commuting invariants in order to assure self-adjointness and to recover the Laplacian for the case of free motion. For each case – in the ambient space $\mathbb{R}^n$, the sphere and the ellipsoid – the Schrödinger equations are completely separated in hyperellipsoidal coordinates, giving equations of generalized Lamé type.

Introduction

A general method for realizing integrable Hamiltonian systems as isospectral flows in rational coadjoint orbits of loop algebras was developed in [AHP, AHH1-AHH4]. This approach begins with a moment map embedding of certain Hamiltonian quotients of symplectic vector spaces into finite dimensional Poisson subspaces of the dual $\widetilde{\mathfrak{gl}}(r)^{+*}$ of the positive frequency part of the loop algebra $\widetilde{\mathfrak{gl}}(r)$ (or certain subalgebras thereof). The Adler–Kostant–Symes (AKS) theorem [A, K, S] then implies that the spectral invariants provide commuting integrals inducing isospectral flows determined by matrix Lax equations. The level sets of these commuting invariants are shown to determine Lagrangian foliations on the rational coadjoint orbits, and hence completely integrable systems. Finally, a special

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