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Ergodicity of the 2-D Navier–Stokes Equation Under Random Perturbations

Franco Flandoli[1], **Bohdan Maslowski**[2]

[1] Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56100 Pisa, Italy
[2] Mathematical Institute, Czech Academy of Sciences, Žitná 25, 11567 Prague 1, Czech Republic

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Abstract: A 2-dimensional Navier–Stokes equation perturbed by a sufficiently distributed white noise is considered. Existence of invariant measures is known from previous works. The aim is to prove uniqueness of the invariant measures, strong law of large numbers, and convergence to equilibrium.

1. Introduction

Consider a viscous incompressible fluid in a bounded domain. The following ergodic principle lies at the foundation of the statistical approach to the fluid dynamic (see [17, 23]): there exists an equilibrium measure μ over the phase space (a space of velocity fields) such that, for every regular observable defined over the phase space, and for every initial velocity field (except for a set of initial fields that is negligible in some sense), the time average of the observable tends, as time goes to infinity, to the mean value of the observable with respect to μ. A rigorous justification of this principle is not known. The aim of this paper is to prove this result in the case of the 2-dimensional Navier–Stokes equation perturbed by a sufficiently distributed white noise. The existence of invariant measure for such equation has been already known (cf. [23, 4, 11, 12], under different conditions). Here it is proved, under proper assumptions on the noise, that the invariant measure is unique, it satisfies a strong law of large numbers, and the convergence to equilibrium takes place.

In the case of finite dimensional differential equations, it is well known that a non-degenerate white noise perturbation yields the previous ergodic results. However, the analysis of this problem in the infinite dimensional case is considerably more difficult, and only the recent development of suitable techniques (cf. for instance [7, 15, 16, 19, 20, 21, 3, 6, 8]) gave the possibility to prove the result for Navier–Stokes equation. Some of the restrictions that we have to impose on the noise are quite standard compared with the current literature on ergodicity of infinite dimensional equations, but it is reasonable to expect that they could be removed by future improvements of the methods.