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Statistical Properties of Shocks in Burgers Turbulence★

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Abstract: We consider the statistical properties of solutions of Burgers' equation in the limit of vanishing viscosity, $\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}(\frac{1}{2}u(x,t)^2) = 0$, with Gaussian white-noise initial data. This system was originally proposed by Burgers[1] as a crude model of hydrodynamic turbulence, and more recently by Zel'dovich *et al.*[12] to describe the evolution of gravitational matter at large spatio-temporal scales, with shocks playing the role of mass clusters. We present here a rigorous proof of the scaling relation $P(s) \propto s^{1/2}, s \ll 1$, where $P(s)$ is the cumulative probability distribution of shock strengths. We also show that the set of spatial locations of shocks is discrete, i.e. has no accumulation points; and establish an upper bound on the tails of the shock-strength distribution, namely $1 - P(s) \leqq \exp\{-Cs^3\}$for $s \gg 1$. Our method draws on a remarkable connection existing between the structure of Burgers turbulence and classical probabilistic work on the convex envelope of Brownian motion and related diffusion processes.

1. Introduction

The study of Burgers' equation with random initial data

$$\begin{cases} \dfrac{\partial u(x,\, t)}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{u(x,\, t)^2}{2}\right) = \nu \dfrac{\partial^2 u(x,\, t)}{\partial x^2}, \\ u(x, t = 0) = u_0(x) \end{cases} \tag{1}$$

where $u_0(x)$ is a Gaussian white noise; i.e. $\langle u_0(x)\rangle = 0$; $\langle u_o(x)u_0(y)\rangle = \delta(x-y)$ originated in the classical work of Burgers[1] as a simplified model of hydrodynamic turbulence. It is now widely recognized that this model, sometimes called "Burgers turbulence" (BT), lacks basic features of Navier-Stokes turbulence such as vorticity stretching, incompressibility, etc.; in fact, the statistical fluctuations of

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