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Stationary States of the Nonlinear Dirac Equation: A Variational Approach

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Abstract: In this paper we prove the existence of stationary solutions of some nonlinear Dirac equations. We do it by using a general variational technique. This enables us to consider nonlinearities which are not necessarily compatible with symmetry reductions.

Section 1. Introduction and Main Results

The nonlinear Dirac equation has been widely used to build relativistic models of extended particles by means of nonlinear Dirac fields. A general form of this equation in the case of an elementary fermion is

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi + \gamma^{0}\nabla F(\psi) = 0\,. \tag{1.1}$$

Here, $\psi : \mathbb{R}^4 \to \mathbb{C}^4, \partial_{\mu}\psi = \frac{\partial}{\partial x^{\mu}}\psi, 0 \leqq \mu \leqq 3$, we have used Einstein's convention for summation over μ, m is a positive constant, $F : \mathbb{C}^4 \to \mathbb{R}$ models a nonlinear interaction and γ^{μ} are the 4×4 Pauli–Dirac matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ and } \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \text{ for } k = 1,2,3 \tag{1.2}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that x^0 plays here the role of time. Throughout this paper we assume that F satisfies $F \in C^2$ and

$$F(e^{i\theta}\psi) = F(\psi) \qquad \text{for all } \theta\,. \tag{1.3}$$

Different functions F have been used to model various types of selfcouplings. For a review on this and historical background see for instance [14].