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Asymptotic Solutions to the Knizhnik–Zamolodchikov Equation and Crystal Base

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Dedicated to the memory of Ansgar Schnizer

Abstract: The Knizhnik–Zamolodchikov equation associated with sl_2 is considered. The transition functions between asymptotic solutions to the Knizhnik–Zamolodchikov equation are described. A connection between asymptotic solutions and the crystal base in the tensor product of modules over the quantum group $U_q sl_2$ is established, in particular, a correspondence between the Bethe vectors of the Gaudin model of an inhomogeneous magnetic chain and the $\mathbb{Q}$-basis of the crystal base.

Introduction

In this work we describe transition functions between asymptotic solutions to the Knizhnik–Zamolodchikov (KZ) equation and establish a connection between asymptotic solutions and the crystal base in the tensor product of modules over a quantum group.

We consider the KZ equation associated with sl_2 and the quantum group $U_q sl_2$, general case can be considered similarly.

For a positive integer m, denote by $L(m)$ the sl_2 irreducible module with highest weight m. For positive integers $m_1, \ldots, m_n$, set $L = L(m_1) \otimes \cdots \otimes L(m_n)$.

Let $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e \in sl_2^{\otimes 2}$ be the Casimir operator. For $i \neq j$ denote by Ω_{ij} the linear operator on L which acts as Ω on the i^{th} and j^{th} factors and as the identity on the other factors. The KZ equation on an L-valued function $\psi(z_1, \ldots, z_n)$ is the system of equations

$$\frac{\partial \psi}{\partial z_j} = \frac{1}{\kappa} \sum_{l \neq j} \frac{\Omega_{jl}}{z_j - z_l} \psi, \quad j = 1, \ldots, n,$$

where κ is a complex parameter. In this paper we assume that κ is not a rational number. The KZ equation is defined over $\mathcal{U}_n = \{z \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$.

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