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Quantum Cohomology of Partial Flag Manifolds $F_{n_1 \cdots n_k}$

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Abstract: We compute the quantum cohomology rings of the partial flag manifolds $F_{n_1 \cdots n_k} = U(n)/(U(n_1) \times \cdots \times U(n_k))$. The inductive computation uses the idea of Givental and Kim [1]. Also we define a notion of the vertical quantum cohomology ring of the algebraic bundle. For the flag bundle $F_{n_1 \cdots n_k}(E)$ associated with the vector bundle E this ring is found.

1. Introduction and Summary

The quantum cohomology rings are only known for some classes of varieties. For Calabi–Yau manifolds one [2] can transform the problem to the "mirror dual" one which deals with variations of the Hodge Structure. The latter is "in principle" solvable, although the real computations in terms of Picard–Fuchs equations may be pretty hard. All known examples describe one- or two-parametric deformations of the classical cohomology rings (see for example [3–5]).

The ring $QH^*(\mathbf{P}^n) = \mathbf{C}[x]/(x^{n+1} - q)$ for the projective spaces has been known since long ago in physics [6] and mathematics (symplectic Floer theory [7]). More generally, one can construct the moduli spaces of rational curves for the toric varieties [8]. Recently Batyrev [9] has conjectured a general formula for the quantum ring in that case. From the physical point of view Batyrev's result can be obtained using the Hamiltonian reduction of the linear σ-model by the real torus [10].

Another example where the hamiltonian reduction of the linear σ-model (by $U(n)$) does work [11, 10] is the Grassmannian $Gr(n, m) = U(n)/(U(n-m) \times U(m))$, because this manifold can be presented as $Gr(n, m) = \mathbf{C}^{nm}//U(m)$. The relations in $QH^*(Gr(n, m))$ were discussed in many physical papers [12–17]. They are proven mathematically [18] for the Grassmannians of 2-planes.

Recently Givental and Kim [1] have computed the quantum cohomology ring of the complete flag manifold $F_n = U(n)/(U(1))^{\times n}$. The main idea of their paper was to use the functoriality properties of the *equivariant* quantum cohomology. Here we extend the arguments of [1] to cover all *partial* flag manifolds $F_{n_1 \cdots n_k} =$