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Distribution of Energy Levels of Quantum Free Particle on the Liouville Surface and Trace Formulae

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Abstract: We consider the Weyl asymptotic formula

$$\#\{E_n \leqq R^2\} = \frac{\mathrm{Area}\, Q}{4\pi} R^2 + n(R)\,,$$

for eigenvalues of the Laplace–Beltrami operator on a two-dimensional torus Q with a Liouville metric which is in a sense the most general case of an integrable metric. We prove that if the surface Q is non-degenerate then the remainder term $n(R)$ has the form $n(R) = R^{1/2}\theta(R)$, where $\theta(R)$ is an almost periodic function of the Besicovitch class B^1, and the Fourier amplitudes and the Fourier frequencies of $\theta(R)$ can be expressed via lengths of closed geodesics on Q and other simple geometric characteristics of these geodesics. We prove then that if the surface Q is generic then the limit distribution of $\theta(R)$ has a density $p(t)$, which is an entire function of t possessing an asymptotics on a real line, $\log p(t) \sim -C_{\pm}t^4$ as $t \to \pm\infty$. An explicit expression for the Fourier transform of $p(t)$ via Fourier amplitudes of $\theta(R)$ is also given. We obtain the analogue of the Guillemin–Duistermaat trace formula for the Liouville surfaces and discuss its accuracy.

1. Introduction

The question about the relation of a quantum system to its classical limit has been discussed since the moment of appearance of quantum mechanics. Recently this question became popular again both in physics and mathematics due to the theory of quantum chaos. It turns out that statistical properties of quantum energy levels depend strongly on the ergodic properties of the underlying classical system. As an important and instructive example one may think of the Laplace–Beltrami operator on a Riemannian manifold and of the geodesic flow in the manifold as its classical counterpart. This case has been widely discussed in many physical and mathematical papers (see, e.g., [1–11, 13–17, 20, 21]).

Let X be a smooth compact Riemannian manifold and let $-\Delta$ be its Laplace–Beltrami operator. The eigen-states of a free quantum particle moving on X are the eigenfunctions of the Laplace–Beltrami operator, i.e., the solutions of the equation