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Non-Local Matrix Generalizations of W-Algebras

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Abstract: There is a standard way to define two symplectic (hamiltonian) structures, the first and second Gelfand–Dikii brackets, on the space of ordinary m^{th}-order linear differential operators $L = -d^m + U_1 d^{m-1} + U_2 d^{m-2} + \cdots + U_m$. In this paper, I consider in detail the case where the U_k are $n \times n$-matrix-valued functions, with particular emphasis on the (more interesting) second Gelfand–Dikii bracket. Of particular interest is the reduction to the symplectic submanifold $U_1 = 0$. This reduction gives rise to matrix generalizations of (the classical version of) the *non-linear* W_m-algebras, called $V_{n,m}$-algebras. The non-commutativity of the matrices leads to *non-local* terms in these $V_{n,m}$-algebras. I show that these algebras contain a conformal Virasoro subalgebra and that combinations W_k of the U_k can be formed that are $n \times n$-matrices of conformally primary fields of spin k, in analogy with the scalar case $n = 1$. In general however, the $V_{m,n}$-algebras have a much richer structure than the W_m-algebras as can be seen on the examples of the *non-linear* and *non-local* Poisson brackets $\{(U_2)_{ab}(\sigma), (U_2)_{cd}(\sigma')\}$, $\{(U_2)_{ab}(\sigma), (W_3)_{cd}(\sigma')\}$ and $\{(W_3)_{ab}(\sigma), (W_3)_{cd}(\sigma')\}$ which I work out explicitly for all m and n. A matrix Miura transformation is derived, mapping these complicated (second Gelfand–Dikii) brackets of the U_k to a set of much simpler Poisson brackets, providing the analogue of the free-field representation of the W_m-algebras.

1. Introduction

Since their discovery by Zamolodchikov [1], W-algebras have been an active field of investigation in theoretical and mathematical physics (see refs. [2, 3] for reviews). They are extensions of the conformal Virasoro algebra by higher spin fields W_k. The commutator of two such higher spin fields is a *local* expression involving *non-linear* differential polynomials of the W_l. W-algebras were found to arise naturally in the context of the $1+1$-dimensional Toda field theories [4] where the higher spin fields

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