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Symplectic Structure of the Moduli Space of Flat Connection on a Riemann Surface

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Abstract: We consider the canonical symplectic structure on the moduli space of flat $\mathscr{G}$-connections on a Riemann surface of genus g with n marked points. For $\mathscr{G}$ being a semisimple Lie algebra we obtain an explicit efficient formula for this symplectic form and prove that it may be represented as a sum of n copies of Kirillov symplectic form on the orbit of dressing transformations in the Poisson–Lie group G^* and g copies of the symplectic structure on the Heisenberg double of the Poisson–Lie group G (the pair (G, G^*) corresponds to the Lie algebra $\mathscr{G}$).

1. Introduction

Being an interesting object of investigations, the moduli space of flat connections on a Riemann surface attracted the attention of many physicists and mathematicians when its relation to the Chern–Simons theory had been discovered [12]. By definition the moduli space (we shall often refer to the moduli space of flat connections in this way) is a quotient of the infinite dimensional space of flat connections over the infinite dimensional gauge group. It is remarkable that this quotient appears to be finite dimensional.

The moduli space $\mathscr{M}$ carries a nondegenerate symplectic structure [3]. It implies the existence of a nondegenerate Poisson bracket on $\mathscr{M}$. Recently the combinatorial description of the moduli space has been suggested [5]. The main idea is to represent the same space $\mathscr{M}$ as a quotient of the finite dimensional space $\mathscr{P}$ over the finite dimensional group action. The Poisson structure has been defined on $\mathscr{P}$ and proved to reproduce the canonical Poisson structure on the moduli space after reduction.

In the first part of this paper we give a combinatorial description of the canonical symplectic structure on $\mathscr{M}$ (see Theorem 1, Sect. 3). This is a bit more natural

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