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Hearing the Zero Locus of a Magnetic Field

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Abstract: We investigate the ground state of a two-dimensional quantum particle in a magnetic field where the field vanishes nondegenerately along a closed curve. We show that the ground state concentrates on this curve as e/h tends to infinity, where e is the charge, and that the ground state energy grows like $(e/h)^{2/3}$. These statements are true for any energy level, the level being fixed as the charge tends to infinity. If the magnitude of the gradient of the magnetic field is a constant b_0 along its zero locus, then we get the precise asymptotics $(e/h)^{2/3}(\mathrm{b}_0)^{2/3}E_* + O(1)$ for every energy level. The constant $E_* \simeq .5698$ is the infimum of the ground state energies $E(\beta)$ of the anharmonic oscillator family $-\frac{d^2}{dy^2} + \left(\frac{1}{2}y^2 - \beta\right)^2$.

1. Introduction

We investigate the asymptotics of a two-dimensional quantum particle in a magnetic field as $\lambda = e/h$ tends to infinity. Here e is its charge and h is Planck's constant. The Hamiltonian of Schrodinger's equation is the covariant Laplacian associated to a connection whose curvature is the magnetic field. Geometrically speaking, the parameter λ is the Chern number, or power, of the line bundle on whose sections this Laplacian acts. If the magnetic field is bounded away from zero by some constant B_0, then it is well-known that the particle's ground state energy $E_1(\lambda)$ satisfies

$$E_1(\lambda) \geqq |\lambda B_0| . \tag{1}$$

We are interested in the case where the field **does vanish.** We will assume that it vanishes along a closed curve C and that its gradient there is nonzero. Our main result is that in this case the ground state concentrates along C as $\lambda \to \infty$ and that its energy satisfies:

$$E_1(\lambda) =_{.} O\left(\lambda^{2/3}\right) . \tag{2}$$

More generally, for every eigenvalue below the continuous spectrum, the same energy bounds and eigensection concentration holds.