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Quantum Cohomology of Flag Manifolds and Toda Lattices

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Abstract: We discuss relations of Vafa's quantum cohomology with Floer's homology theory, introduce equivariant quantum cohomology, formulate some conjectures about its general properties and, on the basis of these conjectures, compute quantum cohomology algebras of the flag manifolds. The answer turns out to coincide with the algebra of regular functions on an invariant lagrangian variety of a Toda lattice.

1. Introduction

Quantum cohomology of compact complex Kahler manifolds was introduced by C. Vafa [V] in connection with the theory of mirror manifolds.

By Vafa's definition, the quantum cohomology $QH^*(X)$ of a compact Kahler manifold X is a certain deformation of the cup-product multiplication in the ordinary cohomology of X. Let a, b, c be three cycles in X representing three given cohomology classes by Poincaré duality. One defines the *quantum cup-product* $a * b$ by specifying its intersection indices with all c. Namely

$$\langle a * b, c\rangle = \sum_{\text{degree } d \text{ discrete holomorphic maps: } (\mathbb{C}P^1,0,1,\infty)\to(X,a,b,c)} \pm q^d .$$

In other words, the intersection index takes in account rational parametrized curves in X with the three marked points – images of 0,1 and ∞ – on the three cycles, a, b and c respectively.

This definition needs some explanations.

1. First of all, a rational curve contributes to the intersection index only if it is "discrete" which means, by definition, that

$$c(d) + \dim X = \operatorname{codim} a + \operatorname{codim} b + \operatorname{codim} c ,$$

where $c(d)$ is the first Chern class c of (the tangent bundle to) X evaluated on the homology class d of the curve, $\dim X$ is the complex dimension

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