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Fractal Drums and the *n*-Dimensional Modified Weyl–Berry Conjecture

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Abstract: In this paper, we study the spectrum of the Dirichlet Laplacian in a bounded (or, more generally, of finite volume) open set $\Omega \in \mathbf{R}^n$ $(n \geqq 1)$ with fractal boundary $\partial\Omega$ of interior Minkowski dimension $\delta \in (n-1, n]$. By means of the technique of tessellation of domains, we give the exact second term of the asymptotic expansion of the "counting function" $N(\lambda)$ (i.e. the number of positive eigenvalues less than λ) as $\lambda \to +\infty$, which is of the form $\lambda^{\delta/2}$ times a negative, bounded and left-continuous function of λ. This explains the reason why the modified Weyl–Berry conjecture does not hold generally for $n \geqq 2$. In addition, we also obtain explicit upper and lower bounds on the second term of $N(\lambda)$.

1. Introduction

Let Ω be an arbitrary non-empty bounded (or, more generally, of finite volume) open set in $\mathbf{R}^n$ $(n \geqq 1)$ with boundary $\partial\Omega$. We consider the following variational eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega\,, \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{P}$$

where Δ denotes the Dirichlet Laplacian in Ω and the problem (P) is to be interpreted in the following sense: we say that the scalar λ is an eigenvalue of (P) if there exists $u \neq 0$ in $H_0^1(\Omega)$ satisfying $-\Delta u = \lambda u$ in the distributional sense.

It is well-known that the spectrum of (P) is discrete and consists of an infinite sequence of positive eigenvalues with finite multiplicity, which may be ordered as

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \cdots \tag{1.1}$$

with $\lambda_k \to +\infty$, as $k \to +\infty$.

We introduce the counting function $N(\lambda)$, which is the number of eigenvalues of (P) less than λ, i.e.

$$N(\lambda) \equiv N(\lambda, -\Delta, \Omega) = \#\{k | \lambda_k < \lambda\}\,. \tag{1.2}$$