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Induced Maps, Markov Extensions and Invariant Measures in One-Dimensional Dynamics

H. Bruin

Department of Pure Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

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Abstract. A way to study ergodic and measure theoretic aspects of interval maps is by means of the Markov extension. This tool, which ties interval maps to the theory of Markov chains, was introduced by Hofbauer and Keller. More generally known are induced maps, i.e. maps that, restricted to an element of an interval partition, coincide with an iterate of the original map.

We will discuss the relation between the Markov extension and induced maps. The main idea is that an induced map of an interval map often appears as a first return map in the Markov extension. For S-unimodal maps, we derive a necessary condition for the existence of invariant probability measures which are absolutely continuous with respect to Lebesgue measure. Two corollaries are given.

1. Statement of the Results

Invariant probability measures are the key ingredient of Birkhoff's Ergodic Theorem, which predicts the statistical behaviour of points in a dynamical system. Especially measures that are absolutely continuous with respect to Lebesgue measure are of interest, because in that case Birkhoff's Ergodic Theorem holds for a large set in Lebesgue sense. We will abbreviate absolutely continuous invariant probability measure to *acip*.

Let us consider interval maps $f : I \to I$ that are piecewise continuous and piecewise monotone. A special case are the unimodal maps, i.e. continuous maps having a unique turning point c. In this paper we concentrate on two methods to construct an acip for f.

First there is the notion of induced map. Let $\{J_i\}_i$ be an interval partition of $J \subset I$ such that $|J \setminus \bigcup_i J_i| = 0$. ($|\ |$ denotes Lebesgue measure). The map $F : \bigcup_i J_i \to J$ is an *induced map* of f, if F coincides on each interval J_i with some iterate f^{s_i} of f. The integer sequence $\{s_i\}_i$ is called the *stopping rule*. F may be constructed

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