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Anderson Localization for the Almost Mathieu Equation: II. Point Spectrum for $\lambda > 2$

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Abstract: We prove that for any $\lambda > 2$ and a.e. ω, θ the pure point spectrum of the almost Mathieu operator $(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda\cos(2\pi(\theta + n\omega))\Psi_n$ contains the essential closure $\hat{\sigma}$ of the spectrum. Corresponding eigenfunctions decay exponentially. The singular continuous component, if it exists, is concentrated on a set of zero measure which is nowhere dense in $\hat{\sigma}$.

1. Introduction

This paper is another attack on the almost-Mathieu operator on $\ell^2(Z)$:

$$(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda\cos(2\pi(\theta + n\omega))\Psi_n \, .$$

This simple-looking operator has been studied extensively for many years. We refer the reader to [1, 2] for a still incomplete list of references. The critical (and physical) value of the coupling constant λ is $\lambda = 2$ (we assume without loss of generality that $\lambda \geqq 0$); it is believed that at $\lambda = 2$ there occurs a transition from pure absolutely continuous to pure point spectrum. The ω here is supposed to be "irrational enough," since for rational ω the potential is periodic and the spectrum is absolutely continuous for all λ, and for Liouville ω (abnormally well approximated by rationals) and $\lambda > 2$ the spectrum of $H(\theta)$ is purely singular continuous [3, 4]. Up to recently the only rigorous reason for this belief was that for $\lambda > 2$ and irrational ω the Lyapunov exponents are positive, which proves the absence of the absolutely continuous part of the spectrum [5, 6]. By Aubry duality there is no pure point spectrum for $\lambda < 2$ [7]. The latest development for any $\lambda < 2$ is the proof of existence of absolutely continuous spectrum that was given by Last [8] for a.e. ω, θ and by Gesztesy and Simon [13] for all ω, θ. Last [8] also proved that for a.e. ω the absolutely continuous spectrum, σ_{ac}, coincides with the spectrum, σ, up to a set of zero Lebesgue measure.

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