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Statistical Mechanics of Nonlinear Wave Equations (4): Cubic Schrödinger

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Abstract: The cubic Schrödinger equation is considered on the circle, both in the de-focussing and the focussing case. The existence of the flow is proved together with the invariance of the appropriate Gibbsian measure, namely the petit canonical measure in the defocussing case and the micro-canonical measure in the focussing case.

1. Introduction

McKean–Vaninsky [1994(1)] discussed the petit canonical resemble for wave equations $\Box Q = \partial^2 Q/\partial t^2 - \partial^2 Q/\partial x^2 = -f(Q)$ of classical type, both on the circle $0 \leqq x < L$ and also for $L \uparrow \infty$. The force $f(Q)$ is odd and of the same signature as Q, *i.e.*, it is a restoring force; also, it is so large that $\int_0^\infty e^{-LF(h)}dh < \infty$ for $F(Q) = \int_0^Q f$. Let[1] $Q^\bullet = P$ and $H = (1/2)\int_0^L [P^2 + (Q')^2] + \int_0^L F(Q)$. Then, with a suitable interpretation of this object, the Gibbsian petit canonical measure $e^{-H}d^\infty P d^\infty Q$ is of total mass $Z < \infty$ and is invariant under the flow $Q^\bullet = P = \partial H/\partial P, P^\bullet = Q'' - f(Q) = -\partial H/\partial Q$ of $\Box Q = -f(Q)$; in particular, the flow exists for almost every choice of data from the petit ensemble.[2]

The present paper deals with nonclassical (dispersive) waves in the special case of the cubic Schrödinger equation:

$$Q^\bullet = -P'' \pm (P^2 + Q^2)P = \partial H/\partial P, \quad P'' = +Q'' \mp (P^2 + Q^2)Q = -\partial H/\partial Q$$

with

$$H = \frac{1}{2}\int_0^L [(P')^2 + (Q')^2]dx \pm \frac{1}{4}\int_0^L (P^2 + Q^2)^2 dx\,.$$

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[1] • signifies $\partial/\partial t$.

[2] Compare Friedlander [1985].