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Positive Lyapunov Exponents for a Class of Deterministic Potentials

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Abstract: Let $V(\theta)$ be a smooth, non-constant function on the torus and let T be a hyperbolic toral automorphism. Consider a discrete one dimensional Schrödinger operator H, whose potential at site j is given by $gV_j = gV(T^j\theta)$. We prove that when $g \geqq 0$ is small and $g^{1/2} \leqq |E| \leqq 2 - g^{1/2}$, the Lyapunov exponent for the cocycle generated by H-E is proportional to g^2. The proof relies on a formula of Pastur and Figotin and on symbolic dynamics.

1. Introduction

In this paper we study the Lyapunov exponent of Schrödinger operators on a one dimensional lattice with an ergodic potential. To define an ergodic potential, let T be an ergodic, measure preserving transformation of a measure space Θ with invariant measure $d\nu$. Let the potential at site n be $V_n(\theta) = V(T^n(\theta))$ where V is a measurable function on Θ. The discrete Schrödinger operator we shall study is given by

$$(H - E)\psi_n = (H(\theta) - E)\psi_n = \psi_{n+1} + \psi_{n-1} + [gV_n(\theta) - E]\psi_n \,. \tag{1}$$

Here $g > 0$ is the coupling constant and ψ is a real valued function on the integer lattice. The Lyapunov exponent defined by

$$\gamma(E) = \lim_{N\to\infty} \frac{\log[|\psi_N|^2 + |\psi_{N+1}|^2]}{2N} \geqq 0 \tag{2}$$

is constant almost everywhere on Θ and independent of the initial values of ψ.

If $g > 0$ and V_j are independent random variables or V comes from a Markov process, positivity of $\gamma(E)$ was proved long ago by Furstenberg [1], Virtser [2], see [3,4] for a review. More generally, Simon [5, b], following the original work by Kotani [5, a] who studied the continuous case, has shown that if the potential of the discrete one-dimensional Schrödinger operator is not determined from values in the past (the class of non-deterministic potentials) then for almost all energies E, the Lyapunov exponent is positive.