Commun. Math. Phys. 168, 39–55 (1995)

Communications in Mathematical Physics

Simultaneous Uniq^{H}ueness of Infinite Clusters in Stationary Random Labeled Graphs

Kenneth S. Alexander[1]

Department of Mathematics, University of Southern California, Los Angeles, CA 90089-1113, USA. Email: alexandr@mtha.usc.edu

Received: 12 October 1993/in revised form: 19 April 1994

Abstract: In processes such as invasion percolation and certain models of continuum percolation, in which a possibly random label $f(b)$ is attached to each bond b of a possibly random graph, percolation models for various values of a parameter r are naturally coupled: one can define a bond b to be occupied at level r if $f(b) \leqq r$. If the labeled graph is stationary, then under the mild additional assumption of positive finite energy, a result of Gandolfi, Keane, and Newman ensures that, in lattice models, for each fixed r at which percolation occurs, the infinite cluster is unique a.s. Analogous results exist for certain continuum models. A unifying framework is given for such fixed-r results, and it is shown that if the site density is finite and the labeled graph has positive finite energy, then with probability one, uniqueness holds simultaneously for all values of r. An example is given to show that when the site density is infinite, positive finite energy does not ensure uniqueness, even for fixed r. In addition, with finite site density but without positive finite energy, one can have fixed-r uniqueness a.s. for each r, yet not have simultaneous uniqueness.

I. Introduction and Statement of Results

There are various models in which percolation processes are naturally coupled for all values of the order parameter. Typically, a value $f(b)$ is attached to each bond b of an infinite graph $(V, \mathscr{B})$ with site set V and bond set $\mathscr{B}$; the graph and/or the values $f(b)$ may be random. A bond b is said to be *occupied at level* r if $f(b) \leqq r$, and one can consider percolation of occupied bonds at various levels r. Some examples follow.

Example 1.1. In invasion percolation, introduced in the mathematical literature in [6], $(V, \mathscr{B})$ is a (nonrandom) lattice in $\mathbb{R}^d$, and the values $\{f(b)\colon b \in \mathscr{B}\}$ are iid uniform in $[0,1]$. The corresponding percolation model is Bernoulli bond percolation.

[1] Research supported by NSF grant DMS-9206139