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The Dirac Operator and Gravitation

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Abstract: We give a brute-force proof of the fact, announced by Alain Connes, that the Wodzicki residue of the inverse square of the Dirac operator is proportional to the Einstein-Hilbert action of general relativity. We show that this also holds for twisted (e.g. by electrodynamics) Dirac operators, and more generally, for Dirac operators pertaining to Clifford connections of general Clifford bundles.

Recently Connes' non-commutative geometry turned out – besides its yielding a fascinating reinterpretation of the full standard model of elementary particles – to be relevant to gravitation. Indeed, on the one hand, Alain Connes made the challenging observation[1] that the Wodzicki[2] residue of the inverse square of the (Atiyah-Singer-Lichnérowicz) Dirac operator yields the Einstein-Hilbert action of general relativity. And moreover he worked out a quantal form of the Polyakov action of strings [2] which reproduces it in the usual case of a Riemann surface, but also makes sense for conformal 4-manifolds, then yielding a conformally invariant action hoped to be connected with gravitation[3].

In this paper we are concerned with the Wodzicki residue of D^{-2}. We first compute this object for the pure Dirac operator (built with the spin connection of a riemannian spin manifold, cf. (1) below): we then obtain, as announed by Connes, a multiple of the scalar curvature (Theorem [1] below). Our proof is a brute-force computation performed in arbitrary coordinate patches.

Now, since the Einstein-Hilbert action and the action of the standard model are both obtained by algorithms based on the Dixmier trace, one naturally wishes to obtain these two actions within a single procedure. Along this line the first natural object to investigate is the Wodzicki residue of $\mathbb{D}^{-2}$, $\mathbb{D}$ the compound Dirac operator

[1] Unpublished, but mentioned verbally in different talks
[2] The Wodzicki residue is a (in fact the unique, thus canonical) trace on the pseudo-differential operators (concentrated on pseudo-differential operators of order – the dimension of the manifold)
[3] The work of the Zürich group on gravitation in non-commutative geometry is based on a different approach related to the Yang-Mills agorithm [6]