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On the Relationship Between Monstrous Moonshine and the Uniqueness of the Moonshine Module

Michael P. Tuite

Department of Mathematical Physics, University College, Galway, Ireland and
Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland.
E-mail: mphtuite@bodkin.ucg.ie

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Abstract. We consider the relationship between the conjectured uniqueness of the Moonshine Module, $\mathscr{V}^\natural$, and Monstrous Moonshine, the genus zero property of the modular invariance group for each Monster group Thompson series. We first discuss a family of possible Z_n meromorphic orbifold constructions of $\mathscr{V}^\natural$ based on automorphisms of the Leech lattice compactified bosonic string. We reproduce the Thompson series for all 51 non-Fricke classes of the Monster group M together with a new relationship between the centralisers of these classes and 51 corresponding Conway group centralisers (generalising a well-known relationship for 5 such classes). Assuming that $\mathscr{V}^\natural$ is unique, we consider meromorphic orbifoldings of $\mathscr{V}^\natural$ and show that Monstrous Moonshine holds if and only Z_r if the only meromorphic orbifoldings of $\mathscr{V}^\natural$ are $\mathscr{V}^\natural$ itself or the Leech theory. This constraint on the meromorphic orbifoldings of $\mathscr{V}^\natural$ therefore relates Monstrous Moonshine to the uniqueness of $\mathscr{V}^\natural$ in a new way.

1. Introduction

The Moonshine Module, $\mathscr{V}^\natural$, of Frenkel, Lepowsky and Meurman (FLM) [1, 2, 3] is historically the first example of a Z_2 orbifold model [4] in Conformal Field Theory (CFT) [5, 6]. The orbifold construction is based on a reflection automorhism of the central charge 24 bosonic string which has been compactified [7] via the Leech lattice cf. [8]. The vertex operators (primary conformal fields) of $\mathscr{V}^\natural$ form a closed meromorphic Operator Product Algebra (OPA) [3, 9, 10] which is preserved by the Fischer-Griess Monster group, M [11]. By construction, $\mathscr{V}^\natural$ has no massless (conformal dimension 1) operators and has modular invariant partition function $J(\tau)$, the unique modular invariant meromorphic function with a simple pole at $\tau = \infty$ and no constant term. $J(\tau)$ is unique because the fundamental region for the full modular group is of genus zero cf. [12]. Conway and Norton [13] conjectured that this genus zero property extends to other modular functions called the Thompson series $T_g(\tau)$ for each conjugacy class of $g \in M$ [14]. Such a genus zero modular function is called