The Twistor Theory of Equations of KdV Type: I

L.J. Mason[1],*, **M.A. Singer[2],****

[1] The Mathematical Institute, 24–29 St Giles, Oxford OX1 3LB, U.K. E-mail: lmason@maths.
oxford.ac.uk
[2] Department of Mathematics and Statistics, James Clerk Maxwell Building, King's Buildings,
University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, U.K. E-mail: msinger@spam.
adelaide.edu.au Fax (Singer) + 618 232 5670

Abstract: This article is the first of two concerned with the development of the theory of equations of KdV type from the point of view of twistor theory and the self-dual Yang–Mills equations. A hierarchy on the self-dual Yang–Mills equations is introduced and it is shown that a certain reduction of this hierarchy is equivalent to the n-generalized KdV-hierarchy. It also emerges that each flow of the n-KdV hierarchy is a reduction of the self-dual Yang–Mills equations with gauge group SL_n. It is further shown that solutions of the self-dual Yang–Mills hierarchy and their reductions arise via a generalized Ward transform from holomorphic vector bundles over a twistor space. Explicit examples of such bundles are given and the Ward transform is implemented to yield a large class of explicit solutions of the n-KdV equations. It is also shown that the construction of Segal and Wilson of solutions of the n-KdV equations from loop groups is contained in our approach as an ansatz for the construction of a class of holomorphic bundles on twistor space.

A summary of the results of the second part of this work appears in the Introduction.

1. Introduction

In the subject "Integrable systems and self-duality" – which has seen much recent activity [Wa, MS, M, MW] – one can identify two clear goals. First, to relate as many integrable systems as possible to the self-duality equations; and secondly, to understand the many techniques for generating solutions of integrable systems in terms of the twistor description of the self-duality equations.

This paper and its sequel are devoted to integrable systems of Korteweg de Vries (KdV) type. As for the first of the above goals, we relate the n-generalized KdV hierarchy to a self-dual Yang–Mills (SDYM) hierarchy and give the corresponding twistor description. As for the second goal, we describe how the twistor description bears on many of the methods associated with these hierarchies: the solutions

---

* SERC Advanced Fellow
** Most of this work was done while Darby Fellow of Mathematics at Lincoln College, Oxford