Commun. Math. Phys. 166, 1 – 26 (1994)

Communications in
Mathematical
Physics

Periodic Nonlinear Schrödinger Equation and Invariant Measures

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Received: 27 October 1993 / in revised form: 21 June 1994

Abstract: In this paper we continue some investigations on the periodic NLSE $iu_t + u_{xx} + u|u|^{p-2} = 0$ $(p \leqq 6)$ started in [LRS]. We prove that the equation is globally wellposed for a set of data ϕ of full normalized Gibbs measure $e^{-\beta H(\phi)} H d\phi(x), H(\phi) = \frac{1}{2}\int |\phi'|^2 - \frac{1}{p}\int |\phi|^p$ (after suitable L^2-truncation). The set and the measure are invariant under the flow. The proof of a similar result for the KdV and modified KdV equations is outlined. The main ingredients used are some estimates from [B1] on periodic NLS and KdV type equations.

1. Introduction

Consider the nonlinear Schrödinger equation (NLSE) in the space periodic setting

$$iu_t + u_{xx} + u|u|^{p-2} = 0\,, \tag{1.1}$$

where u is a function on $\pi \times \mathbb{R}$ on $\pi \times I$ ($I =$ an interval $[0,\tau]$) with an initial condition

$$u(x,0) = \varphi(x)\,, \tag{1.2}$$

where φ is a periodic function of x. Here π stands for the circle, i.e. $\mathbb{R}/\mathbb{Z}$.

In the nonperiodic case (replacing π by $\mathbb{R}$), the Cauchy problem for (1.1)–(1.2) is well understood (see for instance [G-V]). One has a local solution (in the generalized sense) for (1.1) if $p - 2 \leqq \frac{4}{1-2s}$ and data $\varphi \in H^s(\mathbb{R}), s \geqq 0$. The exponent $\frac{4}{1-2s}$ is called H^s-critical (in 1 space dimension). If $p \geqq 6$, there is even for smooth data a possible blow up. In this discussion, the existence result is in fact a global (or local) wellposedness theorem, in the sense of uniqueness and regularity.

In [B1], we have developed a parallel theory in the periodic case, although incomplete so far. The following facts are shown in [B1].

Theorem 1. ($p = 4$) *The cauchy problem*[1]

[1] The result holds both in focusing and defocusing case (with same proof).