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Semiclassical Quantization Rules Near Separatrices

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Abstract: We derive semiclassical quantization equations with uniform estimate of the error term near unstable equilibria of the classical system for the one-dimensional Schrödinger operator.

1. Introduction

The Einstein–Brillouin–Keller (EBK) quantization rule gives semiclassical energy levels of a quantized completely integrable classical system with n degrees of freedom as

$$E(k_1, \ldots, k_n) = H(\pi\hbar(k_1 + (\alpha_1/4)), \ldots, \pi\hbar(k_n + (\alpha_n/4))) , \tag{1.1}$$

where $k_1, \ldots, k_n$ are integer quantum numbers and $\alpha_1, \ldots, \alpha_n$ are the Maslov indices (see, e.g., [BT]). Here $H(I_1, \ldots, I_n)$ is the classical Hamilton function in action-angle coordinates $I_1, \ldots, I_n; \varphi_1, \ldots, \varphi_n$. For a one-dimensional potential hole $U(x)$, (1.1) reduces to the Bohr–Sommerfeld quantization rule of the old quantum mechanics,

$$\hbar^{-1} \int_{x_-}^{x_+} \sqrt{2m(E_k - U(x))}\, dx = \pi(k + (1/2)) . \tag{1.2}$$

The EBK quantization rule has been established in a number of cases (see [CdV, Laz and KMS]). Still many problems remain open and among them the most important problems are, probably, the following two:

(i) Does the EBK quantization rule satisfy the correspondence principle, which means that (1.1) gives all (or at least almost all except finitely many) quantum energy levels?

(ii) What is a uniform quantization rule near separatrices?

In the present paper we address ourselves mostly the second problem in the simplest, one-dimensional case.