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On Functional Determinants of Laplacians in Polygons and Simplicial Complexes

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Abstract: The functional determinant of an elliptic operator with positive, discrete spectrum may be defined as $e^{-Z'(0)}$, where $Z(s)$, the zeta function, is the sum $\sum_n \lambda_n^{-s}$ analytically continued in s. In this paper $Z'(0)$ is calculated for the Laplace operator with Dirichlet boundary conditions inside polygons with the topology of a disc in the Euclidean plane. Our results are complementary to earlier investigations of the determinants on smooth surfaces with smooth boundaries. Our expression can be viewed as the energy for a system of static point particles, corresponding to the corners of the polygon, with self-energy and pair interaction energy. We have completely explicit closed expressions for triangles and regular polygons with an arbitrary number of sides. Among these, there are five special cases (three triangles, the square and the circled), where the $Z'(0)$ are known by other means. One special case fixes an integration constant, and the other provide four independent analytical checks on our calculation.

1. Introduction

One of the basic integrals that arises in many parts in physics is

$$\int_{-\infty}^{\infty} \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^{1/2}} e^{-\frac{1}{2}xAx} = (\mathrm{Det}\,A)^{-1/2}, \tag{1}$$

where A is a real, symmetric matrix with positive eigenvalues. For instance, let (1) describe the integration of fluctuations around a classical solution in imaginary time quantum mechanics, where the Lagrangian has been expanded up to second order. The determinant of A then diverges, and both sides of (1) vanish. The equation is therefore undefined as it stands. As a basic example take a one-dimensional harmonic