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Long-time Asymptotics for Integrable Systems. Higher Order Theory

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Abstract: The authors show how to obtain the full asymptotic expansion for solutions of integrable wave equations to all orders, as $t \to \infty$. The method is rigorous and systematic and does not rely on an a priori ansatz for the form of the solution.

1. Introduction

In [DZ1], the authors introduced a new nonlinear steepest descent-type method for analyzing the asymptotics of oscillatory Riemann–Hilbert (RH) problems. This method has since been used to study rigorously the long-time asymptotics of a wide variety of integrable systems such as the modified Korteweg de Vries (MKdV) equation [DZ1], the nonlinear Schrödinger (NLS) equation [DIZ], the doubly infinite Toda Lattice [K], the autocorrelation function for the transverse Ising chain at critical magnetic field [DZ2], the collisionless shock region for the Korteweg de Vries (KdV) equation [DVZ], and also the Painlevè II equation [DZ3]. In these papers only the leading asymptotics is considered. The purpose of this paper is to show how to obtain the full asymptotic expansion for the solutions in a rigorous and systematic way.

Full asymptotic expansions have been written down in the form of an ansatz for a variety of equations. For example, for NLS

$$iu_t + u_{xx} - 2|u|^2 u = 0\,, \quad u(x,0) = u_0(x) \in S(\mathbb{R})\,, \tag{1.1}$$

Segur and Ablowitz [SA1] introduced the expansion

$$u(x,t) \sim t^{-1/2}\left(\alpha + \sum_{n=1}^{\infty}\sum_{k=0}^{2n} \frac{(\log t)^k}{t^n}\alpha_{nk}\right) e^{ix^2/4t - i\nu\log t}\,, \quad t\to\infty\,, \tag{1.2}$$

where α, α_{nk} and ν are functions of the "slow" variable x/t. The coefficients α_{nk} and the parameter ν can be found explicitly in terms of α via the substitution of (1.2) into