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Asymptotic Stability of Traveling Waves for Scalar Viscous Conservation Laws with Non-convex Nonlinearity

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Abstract: The asymptotic stability of traveling wave solutions with shock profile is considered for scalar viscous conservation laws $u_t+f(u)_x=\mu u_{xx}$ with the initial data u_0 which tend to the constant states $u_\pm$ as $x\to\pm\infty$. Stability theorems are obtained in the absence of the convexity of f and in the allowance of s (shock speed) $=f'(u_\pm)$. Moreover, the rate of asymptotics in time is investigated. For the case $f'(u_+)<s<f'(u_-)$, if the integral of the initial disturbance over $(-\infty, x)$ is small and decays at the algebraic rate as $|x|\to\infty$, then the solution approaches the traveling wave at the corresponding rate as $t\to\infty$. This rate seems to be almost optimal compared with the rate in the case $f=u^2/2$ for which an explicit form of the solution exists. The rate is also obtained in the case $f'(u_\pm)=s$ under some additional conditions. Proofs are given by applying an elementary weighted energy method to the integrated equation of the original one. The selection of the weight plays a crucial role in those procedures.

1. Introduction

We consider the Cauchy problem for scalar viscous conservation laws:

$$u_t+f(u)_x=\mu u_{xx}, \quad x\in\mathbf{R},\ t>0\ , \tag{1.1}$$

$$u(0,x)=u_0(x), \quad x\in\mathbf{R}\ , \tag{1.2}$$

where $f\in C^2$ under consideration, μ is a positive constant and the initial data $u_0(x)$ is asymptotically constant as $x\to\pm\infty$:

$$u_0(x)\to u_\pm \quad \text{as} \quad x\to\pm\infty\ . \tag{1.3}$$

Let Eq. (1.1) admit traveling wave solutions with shock profile such that

$$u=U(x-st)\equiv U(\xi), \quad U(\xi)\to u_\pm \quad \text{as} \quad \xi\to\pm\infty\ , \tag{1.4}$$

where the constants $u_\pm$ and s (shock speed) satisfy the Rankine–Hugoniot condition

$$-s(u_+-u_-)+f(u_+)-f(u_-)=0\ , \tag{1.5}$$