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Anderson Localization for the Almost Mathieu Equation: A Nonperturbative Proof

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Abstract: We prove that for any diophantine rotation angle ω and a.e. phase θ the almost Mathieu operator $(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda \cos(2\pi(\theta + n\omega))\Psi_n$ has pure point spectrum with exponentially decaying eigenfunctions for $\lambda \geqq 15$. We also prove the existence of some pure point spectrum for any $\lambda \geqq 5.4$.

1. Introduction

In this paper we study localization for the almost-Mathieu operator on $\ell^2(\mathbf{Z})$:

$$(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda \cos(2\pi(\theta + n\omega))\Psi_n .$$

The almost-Mathieu operator attracted a lot of interest especially in the last decade. For references before 1985 see [1]. Some of the later references are [2–7].

While it is very well understood and commonly believed that for diophantine ω and $|\lambda| > 2$ the operator $H(\theta)$ should have pure point spectrum with exponentially decaying eigenfunctions for almost every θ, this is not yet rigorously proved. Localization was proved by Sinai [2] and Fröhlich, Spencer and Wittwer [3] in the perturbative regime: $|\lambda|$ "big enough." The methods developed in [2] and [3] are very different but have to overcome one common difficulty: the absence of a Wegner-type estimate that would give control of eigenvalue splitting. In both [2] and [3] the gaps in the spectrum were estimated by special inductive multi-scale procedures and these were the hardest parts of the proofs. The estimation of the gaps is in fact the main difficulty in the proof of localization for any potential that is not random enough to be treated by a Wegner-type argument. In this paper we present a new approach to this difficulty: avoiding rather than fighting it. That makes the proof of localization shorter and more elementary.

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