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Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry

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Abstract: The paper is devoted to the mathematical aspects of topological quantum field theory and its applications to enumerative problems of algebraic geometry. In particular, it contains an axiomatic treatment of Gromov–Witten classes, and a discussion of their properties for Fano varieties. Cohomological Field Theories are defined, and it is proved that tree level theories are determined by their correlation functions. Application to counting rational curves on del Pezzo surfaces and projective spaces are given.

1. Introduction

Let V be a projective algebraic manifold.

Methods of quantum field theory recently led to a prediction of some numerical characteristics of the space of algebraic curves in V, especially of genus zero, eventually endowed with a parametrization and marked points. It turned out that an appropriate generating function Φ whose coefficients are these numbers has a physical meaning ("potential," or "free energy"), and its analytical properties can be guessed with such a precision that it becomes uniquely defined. In particular, when V is a Calabi–Yau manifold, Φ conjecturally describes a variation of Hodge structure of the mirror dual manifold in special coordinates (see contributions in [Y, Ko, Ma2]) which identifies Φ as a specific combination of hypergeometric functions.

In this paper, we use a different tool, the so called "associativity" relations, or WDVV–equations (see [W, D]), in order to show that for Fano manifolds these equations tend to be so strong that they can define Φ uniquely up to a choice of a finite number of constants. (For Calabi–Yau varieties these equations hold as well, but they do not constrain Φ to such extent.)

Mathematically, this formalism is based upon the theory of *the Gromov–Witten classes*. In our setup, they form a collection of linear maps $I^V_{g,n,\beta} : H^*(V,\mathbf{Q})^{\otimes n} \to H^*(\overline{M}_{g,n},\mathbf{Q})$ that ought to be defined for all integers $g \geq 0, n + 2g - 3 \geq 0$, and homology classes $\beta \in H_2(V,\mathbf{Z})$ and are expected to satisfy a series of formal