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Asymptotic Stability of Solitary Waves

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Abstract: We show that the family of solitary waves (1-solitons) of the Korteweg–de Vries equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0 ,$$

is asymptotically stable. Our methods also apply for the solitary waves of a class of generalized Korteweg–de Vries equations,

$$\partial_t u + \partial_x f(u) + \partial_x^3 u = 0 .$$

In particular, we study the case where $f(u) = u^{p+1}/(p+1)$, $p = 1, 2, 3$ (and $3 < p < 4$, for $u > 0$, with $f \in C^4$). The same asymptotic stability result for KdV is also proved for the case $p = 2$ (the modified Korteweg–de Vries equation). We also prove asymptotic stability for the family of solitary waves for all but a finite number of values of p between 3 and 4. (The solitary waves are known to undergo a transition from stability to instability as the parameter p increases beyond the critical value $p = 4$.) The solution is decomposed into a modulating solitary wave, with time-varying speed $c(t)$ and phase $\gamma(t)$ (*bound state part*), and an infinite dimensional perturbation (*radiating part*). The perturbation is shown to decay exponentially in time, in a local sense relative to a frame moving with the solitary wave. As $p \to 4^-$, the local decay or radiation rate decreases due to the presence of a *resonance pole* associated with the linearized evolution equation for solitary wave perturbations.

1. Introduction

Solitary waves are a class of finite energy, spatially localized solutions of nonlinear dispersive partial differential equations of Hamiltonian type. In many such systems, computer simulations and certain analytical results suggest that, in general, solutions