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Renormalization of Random Jacobi Operators

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Abstract: We construct a Cantor set $\mathcal{J}$ of limit-periodic Jacobi operators having the spectrum on the Julia set J of the quadratic map $z \mapsto z^2 + E$ for large negative real numbers E. The density of states of each of these operators is equal to the unique equilibrium measure μ on J. The Jacobi operators in $\mathcal{J}$ are defined over the von Neumann-Kakutani system, a group translation on the compact topological group of dyadic integers. The Cantor set $\mathcal{J}$ is an attractor of the iterated function system built up by the two renormalisation maps $\Phi_\pm : L = \psi(D_\pm^2 + E) \mapsto D_\pm$. To prove the contraction property, we use an explicit interpolation of the Bäcklund transformations by Toda flows. We show that the attractor $\mathcal{J}$ is identical to the hull of the fixed point L_+ of Φ_+.

1. Introduction

Random Jacobi operators are discrete one-dimensional Laplacians and are discrete approximations of one-dimensional random Schrödinger operators. The literature about such operators is huge and a part is by now covered by text books like [CFKS, CL, C, PF].

Dynamical systems obtained by *iteration of rational maps* have a rich structure. Among these systems, the quadratic map $z \mapsto z^2 + E$ is studied best. For reviews in the large literature we refer to [Bla, CG, Ere, M].

Toda differential equations are *integrable Hamiltonian systems* and are discretisations of the Korteweg de Vries systems. According to the chosen boundary condition, the investigation of the Toda systems touches different areas in mathematics. We refer to [FT, Tod, Per, K1].

The subject of this article is located in the intersection of the above three fields. We study random Jacobi operators having the symmetry of being invariant under a

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