Commun. Math. Phys. 164, 1–15 (1994)

Communications in Mathematical Physics

An Explicit Description of the Fundamental Unitary for $SU(2)_q$

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Received: 17 May 1993

Abstract: We give a concrete description of an isometry v from $\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})$ to $\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{Z})$ whose existence has recently been discovered by Woronowicz [11]. The isometry v gives the comultiplication δ on the C*-algebra A of the quantum group $SU(2)_q$ through the formula $\delta(x)=v(x\otimes 1)v^*(x\in A)$, where 1 is the identity operator on $\ell^2(\mathbb{Z}\times\mathbb{Z})$. The matrix entries of v are described in terms of little q-Jacobi polynomials. Using v, we give a concrete description of a unitary operator V on $H_\eta\otimes H_\eta$ such that $(\pi_\eta\otimes\pi_\eta)\delta(x)=V(\pi_\eta(x)\otimes 1)V^*$, where $H_\eta=\ell^2(\mathbb{N}\times\mathbb{Z}\times\mathbb{N})$ and π_η: $A\to L(H_\eta)$ is the GNS representation associated with the Haar state η on A. The operator V satisfies the pentagonal identity of Baaj and Skandalis [1].

1. Introduction

The C*-algebra A of the quantum group $SU(2)_q$, where $0<q<1$, is a unital C*-algebra with generators a, c and relations that make

$$\begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \tag{1.1}$$

a unitary element of $M_2(A)$, namely

$$a^*a+c^*c=aa^*+q^2c^*c=1\ ,$$
$$ac=qca,\quad ac^*=qc^*a,\quad cc^*=c^*c\ . \tag{1.2}$$

There is a natural representation of A on the Hilbert space $\ell^2(\mathbb{N}\times\mathbb{Z})$, which was described by Woronowicz [9] as follows. For any set I, denote the standard orthonormal basis of $\ell^2(I)$ by $\{\varepsilon_i: i\in I\}$; if J is another index set then we identify $\ell^2(I)\otimes\ell^2(J)$ with $\ell^2(I\times J)$ by the correspondence $\varepsilon_i\otimes\varepsilon_j\leftrightarrow\varepsilon_{(i,j)}$ $(i\in I, j\in J)$, and we abbreviate $\varepsilon_{(i,j)}$ to ε_{ij}. Define bounded linear operators a, c on $\ell^2(\mathbb{N}\times\mathbb{Z})$ by

$$a\varepsilon_{ki}=(1-q^{2k})^{1/2}\,\varepsilon_{k-1\,i},\quad c\varepsilon_{ki}=q^k\varepsilon_{k\,i-1}\ (k\in\mathbb{N}, i\in\mathbb{Z})\ . \tag{1.3}$$